INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 3, 23–31 (1983)

INVESTIGATION OF SOLUTION OF NAVIER–STOKES EQUATIONS USING A VARIATIONAL FORMULATION

A. ECER*
Purdue University at Indianapolis, Indianapolis, Indiana 46205, U.S.A.

R. K. ROUT
Structural Dynamics Research Corporation, Milford, Ohio, 45150, U.S.A.

P. WARD
Structural Dynamics Research Corporation, Hitchin, Hertfordshire, U.K.

SUMMARY

A variational formulation for the solution of two dimensional, incompressible viscous flows has been developed by one of the authors.[1] The main objective of the present paper is to demonstrate the applicability of this approach for the solution of practical problems and in particular to investigate the introduction of boundary conditions to the Navier–Stokes equations through a variational formulation. The application of boundary conditions for typical internal and external flow problems is presented. Sample cases include flow around a cylinder and flow through a stepped channel.

Quadrilateral, bilinear isoparametric elements are utilized in the formulation. A single-step, implicit, and fully coupled numerical integration scheme based on the variational principle is employed. Presented results include sample cases with different Reynolds numbers for laminar and turbulent flows. Turbulence is modelled using a simple mixing length model. Numerical results show good agreement with existing solutions.

KEY WORDS Finite Element Navier–Stokes Incompressible Flows

INTRODUCTION

The numerical solution of the Navier–Stokes equations for incompressible, viscous flows requires the coupled solutions of two types of equations. The first equation is the condition of incompressibility:

$$u_x + v_y = 0 \tag{1}$$

This condition is a kinematic constraint on the velocity field (u, v). A second set of equations specifies the conservation of momentum:

$$u_t + uu_x + vu_y = \frac{1}{\rho} p_x + \nu \nabla^2 u \tag{2}$$

$$v_t + uv_x + vv_y = \frac{1}{\rho} p_y + \nu \nabla^2 v \tag{3}$$

These equations are non-linear owing to the presence of the convection terms. The task of obtaining a solution requires the treatment of the coupling between the two types of

* Professor

0271–2091/83/010023–09$01.00

Received 16 August 1981
Revised 5 March 1982

equations. Boundary conditions for u, v and consequently p have to be included consistently in the formulation. In practice, however, one observes that coupling between pressures and velocities produces difficulties both for the coupled numerical integration of the system and the introduction of the boundary conditions.

A second approach to the solution of the Navier–Stokes equations is to introduce vorticity as a new variable and replace equations (2) and (3) with the vorticity transport equation:

$$v_x - u_y = \zeta \tag{4}$$

$$\zeta_t + u\zeta_x + v\zeta_y = \nu(\zeta_{xx} + \zeta_{yy}) \tag{5}$$

In this case, pressure is eliminated as a variable. A new kinematic relationship, as defined in equation (4), is introduced and only one vorticity transport equation has to be integrated. Now, the problem reduces to one of obtaining a coupled solution of the boundary conditions. In practice, again difficulties are encountered in obtaining full coupling between these equations, especially at the boundaries.

In this paper, the solution of the Navier–Stokes equations is discussed based on a variational form. Using a Clebsch type of transformation:

$$u = \phi_x + \beta_y - \eta\zeta_x, \tag{6}$$

$$v = \phi_y - \beta_x - \eta\zeta_y, \tag{7}$$

a variational formulation for the solution of equations (1), (4) and (5) was presented in Reference 1 which also introduces the variables ϕ, β and η. The details of the formulation are not repeated here. However, the importance of the proper treatment of the boundary conditions is presented in detail.

METHOD OF SOLUTION

A variational functional can be defined for the solution of the Navier–Stokes equations (1), (4) and (5) in terms of the basic variables (ϕ, β, ζ, η) as defined in equations (6) and (7). The Lagrangian, L, for this case can be written as follows:

$$L = \tfrac{1}{2}(\phi_x + \beta_y - \eta\zeta_x)^2 + \tfrac{1}{2}(\phi_y - \beta_x - \eta\zeta_y)^2 - \eta\zeta_t - \beta\zeta + \nu\eta\nabla^2\zeta \tag{8}$$

From the above Lagrangian, one can proceed to show (1) that the minimization of the variational functional satisfies equations (1), (4) and (5) with the following natural boundary conditions on the boundary:

$$\int_C (u_n - p)\delta\phi \, \mathrm{d}C = 0 \tag{9}$$

$$\int_C (u_t - q)\delta\phi \, \mathrm{d}C = 0 \tag{10}$$

$$\int_C \eta^2\left[u_n\frac{(\delta\zeta)}{\eta} - \nu\frac{\partial}{\partial n}\frac{\delta\zeta}{\eta}\right]\mathrm{d}C \tag{11}$$

where n and t are defined in the normal and tangential directions to the boundary, C, respectively, and p and q are specified values of normal and tangential velocity on the boundary, C.

Bilinear, four-noded isoparametric elements were employed in the formulation. At each node of the element four varibles (ϕ_i, β_i, ζ_i, η_i) were specified as unknowns. The resulting set of implicit non-linear equations are written in matrix form as:

$$\mathbf{K}^{n+1}\boldsymbol{\phi}^{n+1}=\mathbf{f}^{n+1} \tag{12}$$

where $\mathbf{K}$ is an unsymmetric matrix and is a non-linear function of the nodal variables. These algebraic equations are solved in a fully coupled fashion. The vector $\mathbf{f}$ includes the effects of the specified natural boundary conditions, p and q. To improve the efficiency of the solution scheme a constant coefficient matrix was employed in the following form:[2]

$$\mathbf{K}^{0}\tilde{\boldsymbol{\phi}}^{n+1}=\mathbf{f}^{n+1}+(\mathbf{K}^{0}-\mathbf{K}^{n+1})\boldsymbol{\phi}^{n} \tag{13}$$

with

$$\boldsymbol{\phi}^{n+1}=\omega\tilde{\boldsymbol{\phi}}^{n+1}+(1-\omega)\boldsymbol{\phi}^{n} \tag{14}$$

where $\mathbf{K}^{0}$ is the coefficient matrix formulated at the first time step and ω is a relaxation parameter.

The main advantages of the above formulation can be summarized in two parts:

(a) The resulting formulation produces a fully coupled system for the solution of the equations.
(b) Most of the flow boundary conditions become natural boundary conditions.

In order to illustrate these advantages two sample problems were investigated using the finite element grids shown in Figures 1(a) and 1(b). In the case of two-dimensional, symmetric flow around a cylinder shown in Figure 1a, the boundary conditions can be specified as follows:

On the free stream boundaries (D–E–F–A), u and v are specified as natural boundary conditions.

On the line of symmetry (A–B, C–D), $v=0$ is specified as a natural boundary condition. ζ is specified as a forced boundary condition. β is specified as a forced boundary condition and then u on this line is calculated automatically.

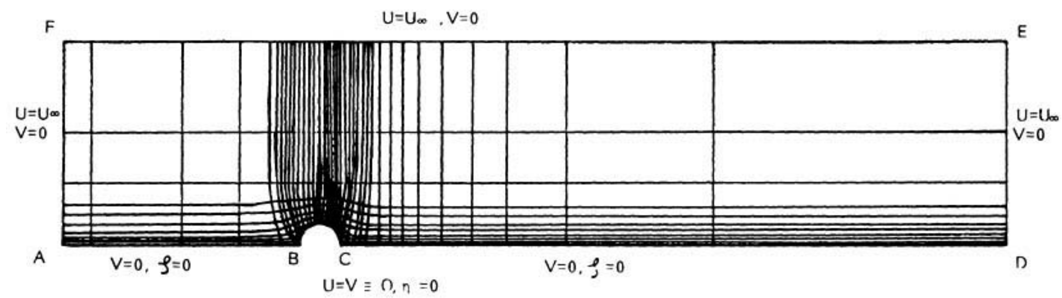

Figure 1(a)

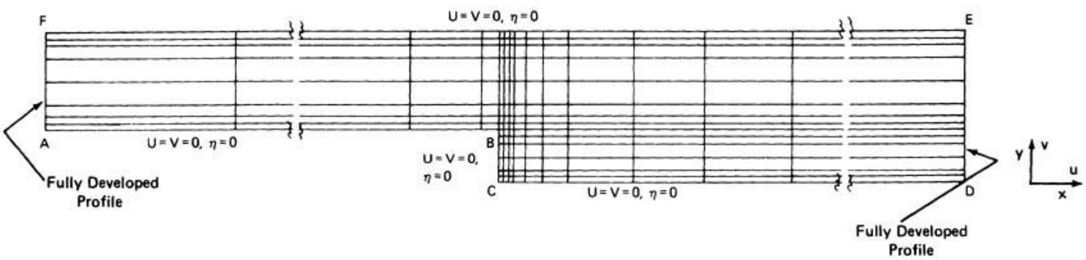

Figure 1(b). Finite element model for step duct

On the cylinder surface (B–C), $u = v = 0$ are specified as natural boundary conditions. Also, $\eta = 0$ is specified as a forced boundary condition. From equation (11), this corresponds to defining the rigid wall as a vorticity generating surface. Vorticity is automatically calculated from the kinematic relationship given in equation (4).

For the second sample problem of a channel with a backward facing step, inlet and outlet boundary conditions are specified as natural boundary conditions for velocities u and v. Vorticity specification is not required on these boundaries. Along the solid walls of the channel (A–B–C–D, and E–F), natural boundary conditions for ($u = v = 0$) and the vorticity generating boundary condition ($\eta = 0$) are specified.

The above examples illustrate the practicality of the present approach for coupling vorticity and velocities. As stated in the introduction previously, this has been a major difficulty in the solution of the Navier–Stokes equations. While only two of the three basic variables (u, v, ζ) are needed to specify the boundary conditions uniquely, the coupled form of the governing equations, as derived from the variational form, provides the necessary coupling to determine the third variable.

MODELLING OF THE TURBULENT FLOW

A simple model was defined to account for the turbulence in the flow based on a mixing length theory.[3] The eddy viscosity near the wall region is defined as:

$$\mu_T = \rho l_\rho^2 \left| \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right| \tag{15}$$

where

$$l_\rho = 0{\cdot}4y\left\{1 - \exp\left(-y\sqrt{\left[\left|\frac{\rho}{\mu}\frac{\partial u}{\partial y}\right|_w \Big/ 26\right]}\right)\right\} \tag{16}$$

until the following value of l_ρ is reached and subsequently used:

$$l_\rho = 0{\cdot}07\delta \tag{17}$$

where δ is an arbitrary cutoff criterion based on the vorticity. For the wake region, the eddy viscosity is calculated from the following equation:

$$\mu_T = 0{\cdot}001176\rho\delta\,|u_\delta - u_{\mathrm{CL}}| \tag{18}$$

where u_δ and u_{CL} are the velocities at the edge of the wake and its centreline, respectively.

DISCUSSION OF RESULTS

Several test cases for the solution of the Navier–Stokes equations were analysed based on the formulation presented above. Results were obtained from the flow around a cylinder and for the flow through a stepped channel using the finite element grids shown in Figures 1(a) and 1(b). these cases are typical examples of external and internal flow problems with separation. The present results were compared with existing numerical and experimental results.

Flow around a cylinder

Flow around a cylinder for $Re = 200$ was calculated using the developed numerical procedure. In this case, the Reynolds number is defined as $Re = U_\infty D/\nu$, where U_∞ is the free stream velocity, D is the diameter of the cylinder and ν is the kinematic viscosity. $\Delta t U_\infty/D = 0{\cdot}0125$ and was employed for numerical integration of the equations. The obtained numerical results are shown in Figures 2 (a) and (b).

Flow in a stepped-channel

Flow in a stepped-channel was analysed using the finite element grid shown in Figure 1(b). Fourteen elements were placed in the flow direction. Eight elements were placed across the duct upstream of the step and fourteen elements were placed downstream of the step as shown in the figure. Both laminar and turbulent flow cases were investigated for this problem. The flow Reynolds number is given as $Re = U_0 h/\nu$, where U_0 is the average velocity at the inlet and h is the step height. Laminar flow cases include $Re = 25$, 73 and 229. Turbulent flow calculations were obtained for $Re = 3025$. For the last three test cases, the grid was extended as shown in Table I, while the number of elements were kept the same. The velocity profiles for all of the above test cases are presented in Figures 3 (a)–(d) and compared with results by others.[4–6] Also, in Figure 4, variation of the length of the separated region is shown as a function of the Reynolds number and compared with previously published results. The presented results show good overall agreement. The time steps and the number of steps to obtain convergence are also summarized in Table I for flows with different Reynolds numbers.

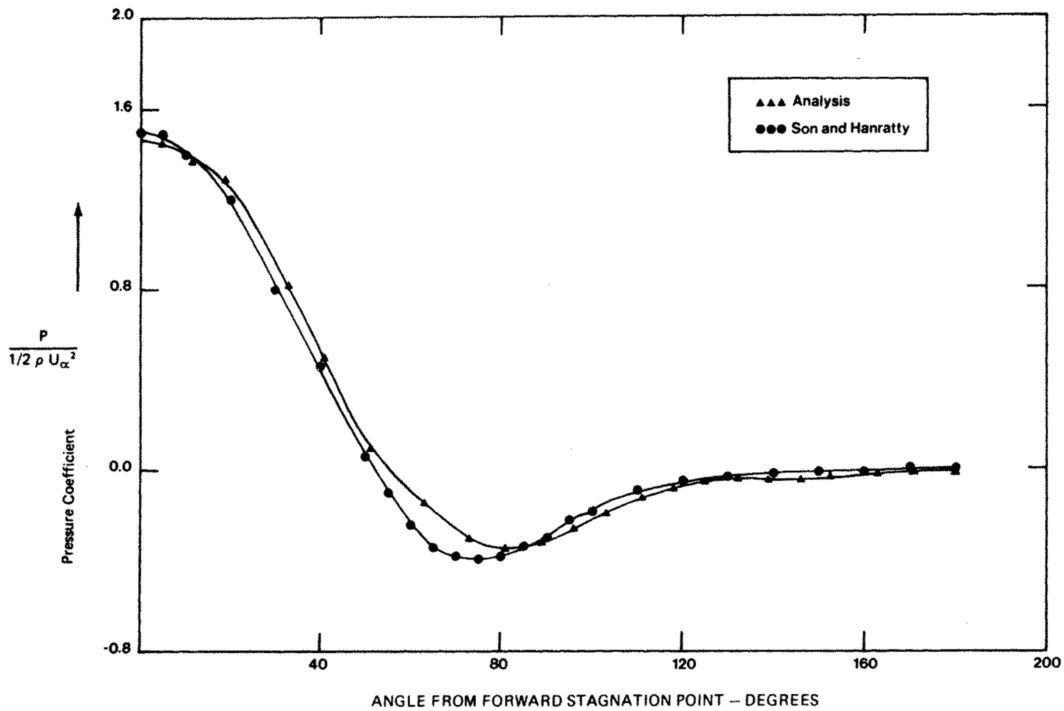

Figure 2(a).

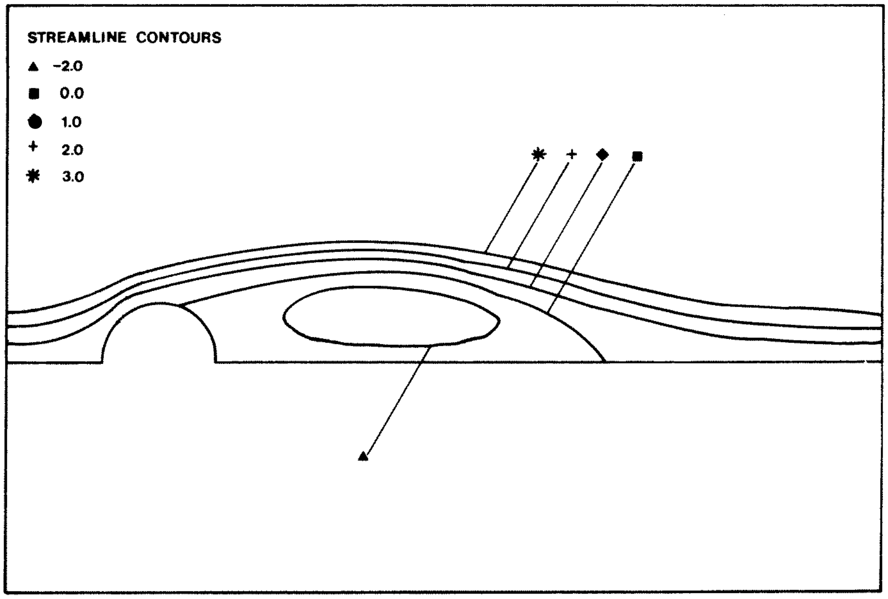

Figure 2(b). Streamline pattern for circular cylinder ($Re = 200$)

Table I. Number of iterations for converged solutions for different Reynolds numbers

Reynolds	Δt	Number of Iterations	Inlet Length	Outlet Length
25	0·1	125	−25·0	+25·0
73	0·1	430	−70·0	+70·0
229	0·01	960	−70·0	+70·0
3025	0·01	30*	−70·0	+70·0

* Solution of $Re = 299$ is used as an initial distribution of $Re = 3025$.

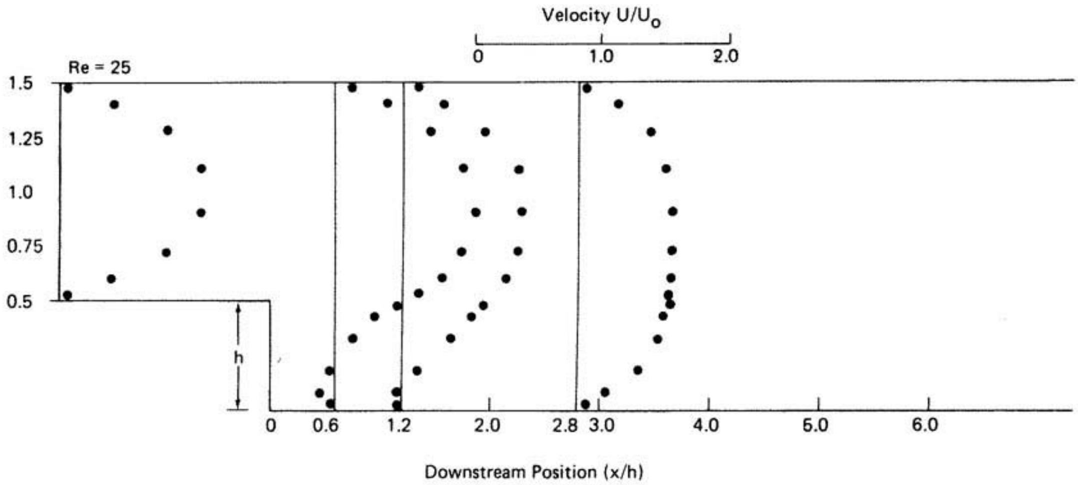

Figure 3(a). Axial velocity profile for step duct (laminar flow)

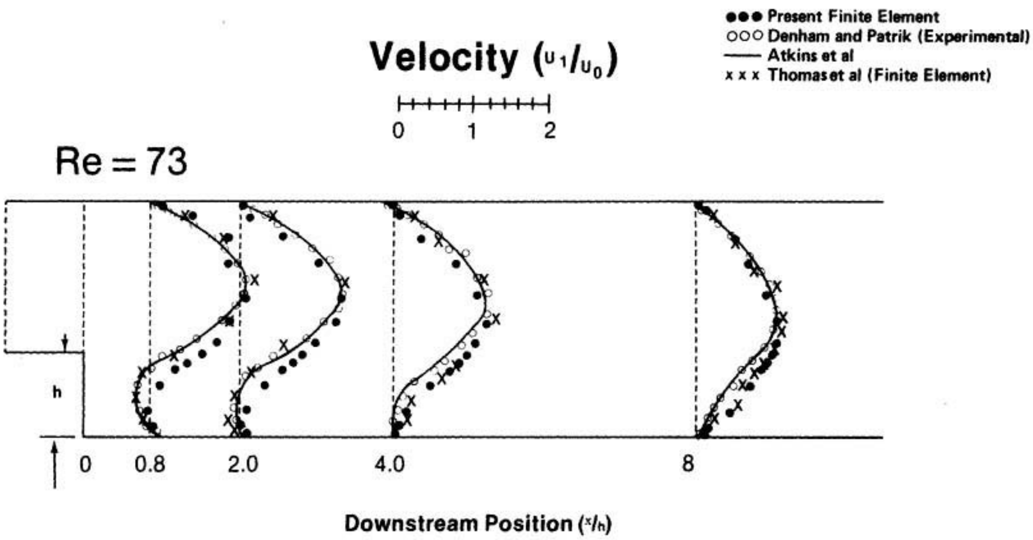

Figure 3(b). Axial velocity profile for step duct (laminar flow)

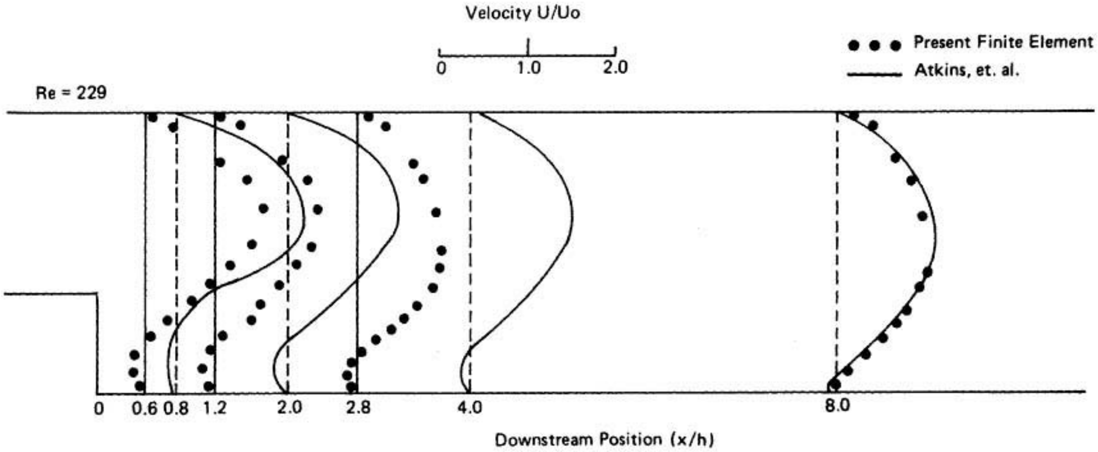

Figure 3(c). Axial velocity profile for step duct (laminar flow)

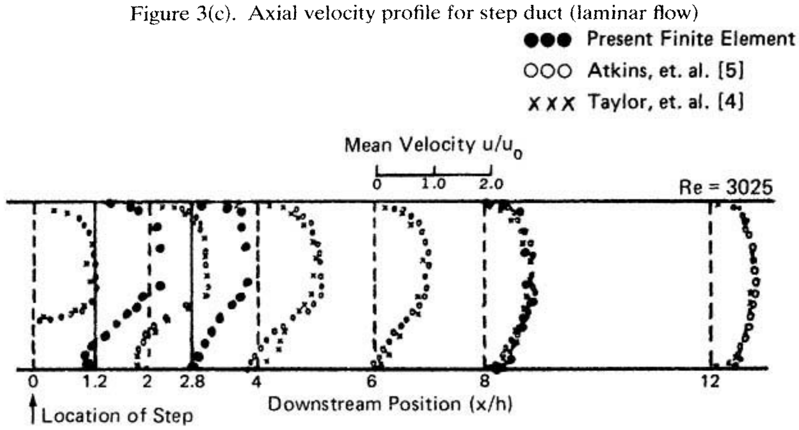

Figure 3(d). Axial velocity profile for step duct (turbulane flow)

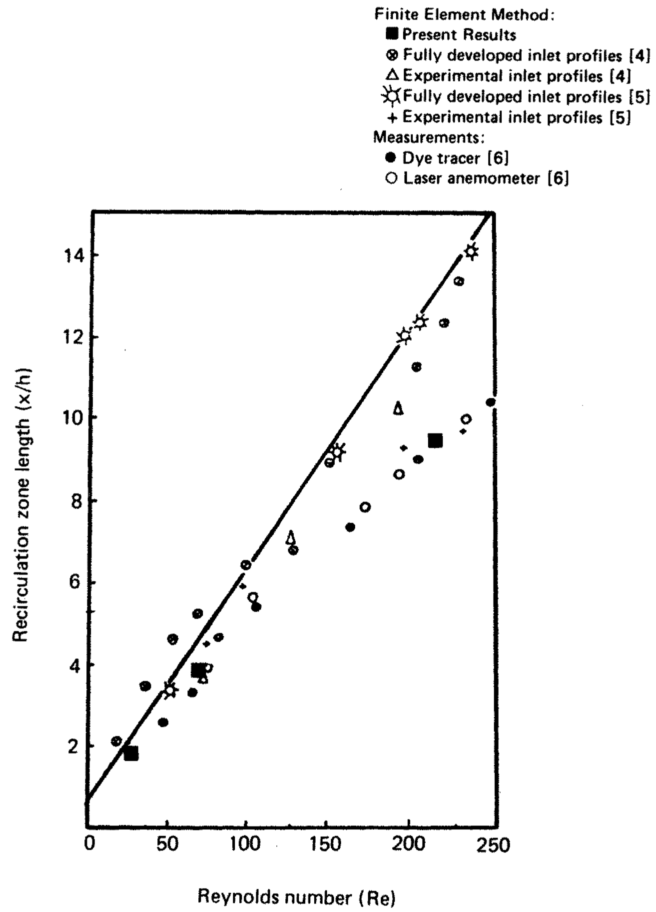

Figure 4. Variation of separation length with flow Reynolds number (laminar)

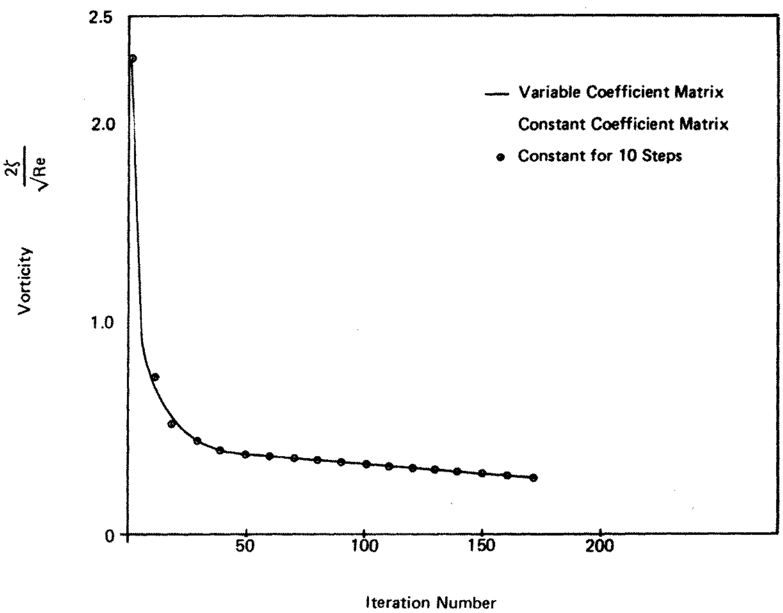

Figure 5. Comparison of convergence of constant and variable coefficient matrix ($Re = 73$)

Convergence characteristics with the constant coefficient matrix

One numerical experiment was performed to determine the convergence characteristics of a solution scheme in which the coefficient matrix was updated periodically. This drastically improves the most costly part of a step, namely the decomposition of the global coefficient matrix.

Convergence results are shown in Figure 5. The inlet wall vorticity for $Re = 73$ in the stepped channel example is plotted as a function of the number of time steps. Results for which the coefficient matrix was decomposed at every step are shown as a solid line.

It can be seen from Figure 5 that the results obtained by decomposing the coefficient matrix every step and every ten steps show negligible difference.

REFERENCES

1. A. Ecer, 'Variational formulation of viscous flows', *Int. J. Num. Meth Eng,* **15,** 1355–1361 (1980).
2. H. U. Akay and A. Ecer, 'Transonic flow computations in cascades using finite element methods', *ASME 26th International Gas Turbine Conference,* Houston, Texas, 8–12 March 1981, to be published in *ASME Journal of Fluids.*
3. G. S. Deiwert, 'Numerical solution of high Reynolds number transonic flows', *AIAA Paper 74–603,* presented at *AIAA 7th Fluid and Plazma Dynamics Conference in Palo Alto, 17–19* June 1974.
4. C. Taylor, C. E. Thomas and K. Morgan, 'Confined turbulent flow utilising the finite element method', T. J. R. Hughes (ed.), *Finite Element Methods for Convection Dominated Flows, AMD* **34,** *ASME,* 213–224 (1979).
5. D. J. Atkins, S. J. Maskel and M. A. Patrick, 'Numerical predictions of separated flows', *Int. J. Num. Meth Eng.* **15,** 129–144 (1980).
6. M. K. Denham and M. A. Patrick, 'Laminar flow over a downstream-facing step in a two-dimensional flow channel', *Trans. Inst. Chem. Engrs.,* **52**(4), 361–367 (1974).
7. J. S. Son and J. T. Hanratty, 'Numerical solution for the flow around, a cylinder at Reynolds numbers of 40, 200 and 500', *J. Fluid Mech.* **35,** (2), 369–386 (1969).